Galaxies Lec 2

Stellar Dynamics

Strong Encounters

strong close encounter: star gets close enough to other star for interaction to dramatically alter its speed and direction.

Potential Energy == Kinetic energy

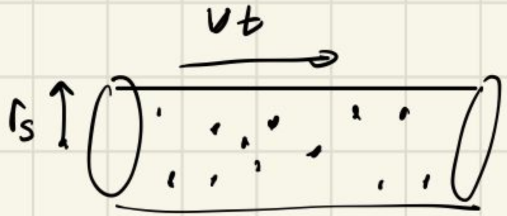
Strong encounters.

① $\Delta E_K \sim \frac{G m^2}{r}$

$\frac{1}{2} m v^2 \sim \frac{G m^2}{r}$

$\boxed{r_s \sim \frac{G m}{v^2}}$ strong encounter radius

②



Probability = 1 = $n \cdot \pi r_s^2 v t$

$\boxed{t = \frac{1}{2\pi n r_s^2}}$

Cannot have strong encounters! Too rare!

Weak Encounters

Remember that

cos(theta) = b/(1+(vt/b)^2)^1/2

Relaxation time is:

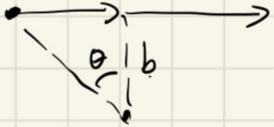
<δv⊥2> = v^2.

Similar to that of kinetic energy

On the order of 1 billion yrs

spread = Initial

Weak encounters $\Rightarrow$ relax when $\langle \Delta v^2 \rangle = v^2$



$$F_\perp = \frac{Gmm}{b}\left(1+\left(\frac{vt}{b}\right)^2\right)^{-3/2} = m\frac{dv}{dt}$$

$$\langle \delta v^2 \rangle = \frac{1}{m}\int F_\perp \, dt$$

Solve for t! $= \underline{10^9\,yrs}$

$$\langle \delta v^2 \rangle = \int_{min}^{max} n v t \left(\frac{2Gm}{bv}\right)^2 2\pi b\, db$$

$$\langle \delta v^2 \rangle = v^2 = \frac{8\pi G^2 m^2 n t}{v} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

Potentials and Motions Under Gravity

Equation of motion

$$\frac{d^2 r}{dt^2} = -\nabla\phi(\vec{r})$$

Poisson equation

$$\nabla^2\phi = 4\pi G\rho(\vec{r})$$

Point mass

- **Example 1.** The Point Mass. Consider a point a distance r from a point mass of M.
    - The gravitational potential is given by

$$\phi(r) = -\frac{GM}{r}$$

$$a = -\nabla\phi$$

$$a = GM/r^2$$

- The circular speed (v_c) of a test particle in orbit at distance r from point mass is

$$a = v_c^2 / r$$

$$v_c^2 = GM/r$$

- We get Keplerian orbits with $\boxed{v_c \propto r^{-1/2}}$

Isothermal Sphere

$$\Phi_{SIS}(r) = V_H^2 \ln(r/r_0)$$

where r_0 = constant and v_H = constant

$$\boxed{V_H^2 = 4\pi G r_0^2 \rho(r_0)}$$

- How does density depend on r for this potential?

$$\rho_{SIS}(r) = \frac{\rho(r_0)}{(r/r_0)^2}$$

- How does the circular velocity depend on r?

$v_c^2 / r = d\phi/dr$

v_c = constant with r (similar to Milky Way rotation curve at large radii, often called a 'dark halo' potential).

Homogeneous Sphere

- **Example 3.** Homogenous sphere with density ρ
  - If the density is some constant ρ, we have $M(r) = (4/3)\pi r^3\rho$
  - The centripetal acceleration is related to the mass interior to radius r.

$$v_c^2 = GM(r)/r$$

$$v_c = [4\pi G\rho/3]^{1/2}\, r$$

  - In this case, circular speed rises linearly with distance r.
  - The orbital period for a mass on a circular orbit is

$$T = \frac{2\pi r}{v_c} = \sqrt{\frac{3\pi}{G\rho}}$$

  - We define the timescale $\sim(G\rho)^{-1/2}$ is the **crossing time,** also known as the **dynamical time**. It gives the characteristic time associated with orbital motion of a star.

→Higher density = shorter crossing/dynamical times.

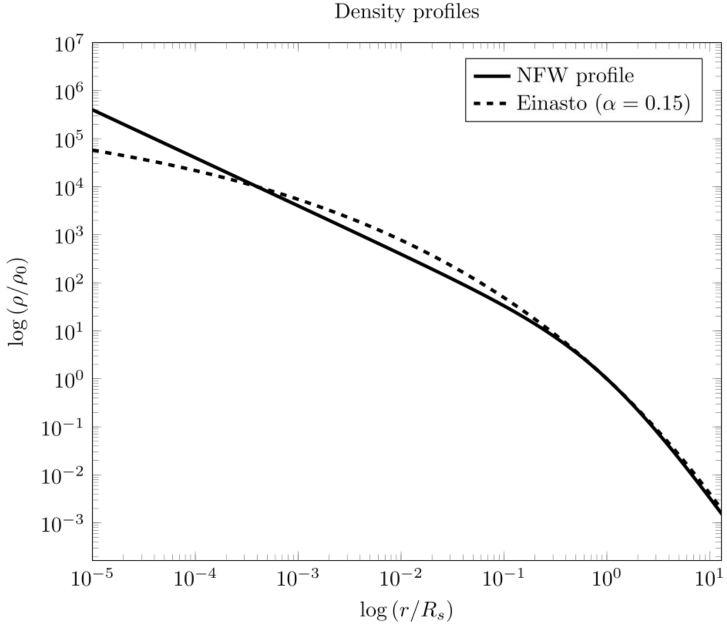
→Processes that involve large portions of a galaxy (i.e., a starburst) happen on roughly timescale, since gravity cannot move material faster through the galaxy. Timescale for gravity to change motion

→Similar to **freefall time**: time cloud of density ρ collapses under its gravity in absence of pressure.

NFW potential

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2}$$

which we call the NFW profile



NFW Is effectively a double power law in density

An inner part of r^-1 and a outer r^-3

In Class Problem

In Class Problems:

$$t_{\rm relax} \approx \frac{2\times10^9\,{\rm yr}}{\ln\Lambda}\left(\frac{V}{10\,{\rm km\,s^{-1}}}\right)^3\left(\frac{m}{\mathcal{M}_\odot}\right)^{-2}\left(\frac{n}{10^3\,{\rm pc^{-3}}}\right)^{-1}$$

Table 3.1 Dynamical quantities for globular and open clusters in the Milky Way

Cluster		σ_r (km s^{-1})	$\log_{10}\rho_c$ ($\mathcal{M}_\odot$ pc^{-3})	r_c (pc)	$t_{\rm relax,c}$ (Myr)	*Mass* ($10^3\mathcal{M}_\odot$)	$\mathcal{M}/L_V$ ($\mathcal{M}_\odot/L_\odot$)
NGC 5139	ω Cen	20	3.1	4	5000	2600	2.5
NGC 104	47 Tuc	11	4.9	0.7	50	800	1.5
NGC 7078	M15	12	>7	<0.1	<1	900	2
NGC 6341	M92	5	5.2	0.5	2	200	1
NGC 6121	M4	4	4–5	0.5	30	60	1
	Pal 13	~0.8	2	1.7	10	3	3–7
NGC 1049	Fornax 3	9	3.5	1.6	600	400	~3
Open cluster	Pleiades	0.5	0.5	3	100	0.8	0.2

Note: σ_r is the dispersion in radial velocity V_r in the cluster core; ρ_c is central density; $t_{\rm relax,c}$ is the relaxation time at the cluster's center found using Equation 3.55 with $V = \sqrt{3}\sigma_r$, $\langle m_*\rangle = 0.3\mathcal{M}_\odot$, and $\Lambda = r_c/1$ AU. Clusters with upper limits to r_c probably have collapsed cores.

- 1. Using the table at left, derive the relaxation time for the center of the globular cluster 47 Tuc, assuming an average stellar mass of 0.5 $M_\odot$.

  Is relaxation more likely to be important in globular clusters than in the disk of Milky Way? Why/why not?

- 2. Compute the relaxation time for the open cluster Pleiades.

  Compare your answer to the crossing time in the Pleiades, defined as ~$2r_c/\sigma_r$.

  How many crossing times does a star finish before relaxation is important?

- 3. Show that a homogenous sphere potential with constant ρ has an equation of motion of a harmonic oscillator. What is the angular frequency? How is this related to the dynamical time?

Virial Theorem

Virial Theorem

$$K = -\frac{1}{2} W$$

Potential

$$W = -\alpha_w \frac{GM^2}{r_h}$$

Velo dispersion

$$\langle v^2 \rangle = 3\sigma_r^2$$

Equate kinetic to get
Virial mass

$$M = \frac{\langle v^2 \rangle r_h}{\alpha_w G}$$

Conserved Quantities

Eq of Motion

$$\frac{d^2\vec{r}}{dt^2} = -\frac{d\phi}{dr}\hat{r}$$

Cross both sides

$$\vec{r} \times \frac{d^2\vec{r}}{dt} = 0$$

(note RHS ~ 0 because both vectors oriented along **r**)

Conserved quantities have **no explicit time dependence** and is constant as a test mass moves along its orbit.

Angular MomentuM

This quantity would have no explicit time dependence thus CONSERVED!

$$\vec{r} \times \frac{d\vec{r}}{dt} = \vec{J}$$

Eq of Motion: polar coordinates
Spherical symmetric potential

$$\frac{d^2\vec{r}}{dt^2} = \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r}$$

Computing the angular momentum

$$r^2\frac{d\theta}{dt} = j = \text{constant}$$

Replace!

$$\frac{d^2\vec{r}}{dt^2} = \left[\frac{d^2 r}{dt^2} - \frac{j^2}{r^3}\right]\hat{r}$$

New Eq motion

$$\frac{d^2 r}{dt^2} - \frac{j^2}{r^3} = -\frac{d\Phi}{dr}$$

In Class Problem

- You measure σ_r ~ 10 km/s for a random sample of stars in a globular cluster. The profile is well fit by a Plummer potential Use the virial theorem to compute its mass.

Remember we need to integrate to get total Potential energy with the density

Get **density** first from poisson's eq

Then integrate with the **potential**

Then apply **virials**

Plummer potential

$$\rightarrow \phi = -\frac{GM}{\sqrt{r^2+a_p^2}}$$

① get density

$$\nabla^2\phi = 4\pi G\rho$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{3a_p^2 M}{(r^2+a_p^2)^{5/2}}$$

$$\rho = \frac{3a_p^2 M}{4\pi\,(r^2+a_p^2)^{5/2}}$$

$$\Omega = \int \phi\rho\,dr = -\frac{3\pi}{32}\frac{GM^2}{a_p} = -2E_k$$

$$-\frac{3\pi}{32}\frac{GM^2}{a_p} = Mv^2$$

$$M \approx \sqrt{\frac{v^2 a_p}{G}}$$

Spherical Potential

Be able to derive the velocities of circular motion along these

- What spherical potentials have we introduced?
  - Uniform density sphere. $\rho \propto$ constant, $v_c \propto r$ (within sphere)
  - Singular isothermal sphere. $\rho \propto r^{-2}$, $v_c \propto$ constant with r
  - Plummer sphere, $\rho \propto$ constant for $r \ll a_p$. $\rho \propto r^{-5}$ for $r \gg a_p$. $\rightarrow v_c \propto r$ for $r \ll a_p$. $v_c \propto r^{-1/2}$ at $r \gg a_p$

ISS

$$\phi = V_H^2 \ln\left(\frac{r}{r_0}\right)$$

$$F = -\frac{d\phi}{dr}$$

$$F = v_s^2 \frac{1}{r} = \frac{v^2}{r}$$

$$v_s = v_H$$

Effective Potentials

Remember the conserved quantity

$$\frac{d^2R}{dt^2} = -\frac{\partial \Phi}{\partial R} + R\left(\frac{d\theta}{dt}\right)^2 = -\frac{\partial \Phi}{\partial R} + \frac{j_z^2}{R^3}$$

plugging in equation for j_z

We can sub in a effective potential

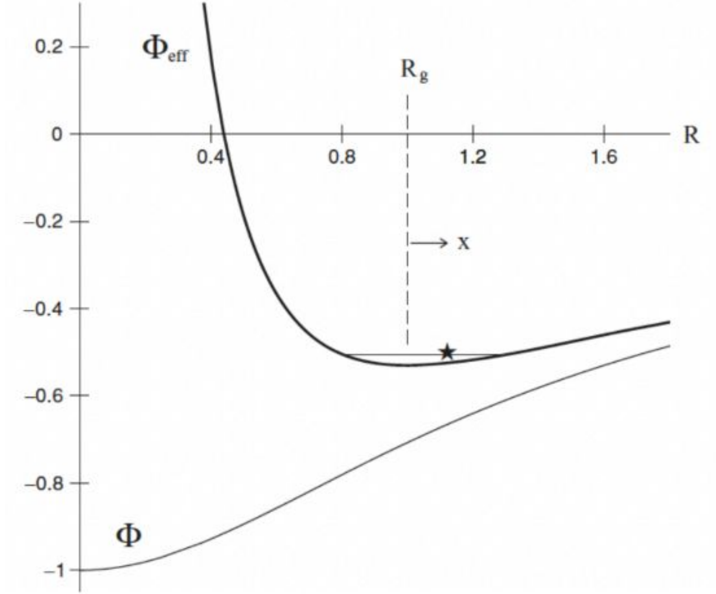
$$\frac{d^2R}{dt^2} = -\frac{\partial \Phi_{eff}}{\partial R}$$

$$\frac{d^2z}{dt^2} = -\frac{\partial \Phi}{\partial z} = -\frac{\partial \Phi_{eff}}{\partial z}$$

$$\boxed{\Phi_{eff}(R,z) = \Phi(R,z) + \frac{j_z^2}{2R^2}}$$

(since centrifugal term has no z-dependence)

If we find the stable place it is the circular. That is the circular radius.



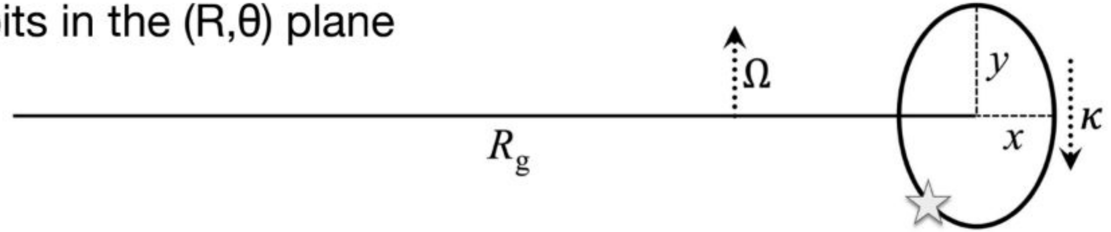
We can solve for this radius

$$\Omega = v_{circ}/R_g = j_z / R_g^2$$

We can solve for this in terms of angular momentum

$$v_{circ}^2 = \left[R\frac{\partial\Phi}{\partial R}\right] = \left(\frac{j_z}{R_g}\right)^2$$

- We now turn to the motion of orbits in the (R,θ) plane



- We will adopt a reference frame moving with angular speed $\Omega = v_{circ}/R_g = j_z / R_g^2$

The circular motion around the ring, act like a perturbed higher order motion.

Asymmetric drift

Asymmetric Drift

When we have two points A (inner) and B (outer) and when viewing the stars they have some epicyclic motion about their guiding centers.

At B (To reach us, they must be at the pericenter (closest point) of their orbits. At pericenter, stars are moving faster than their guiding center)

At A (To reach us at they must be at the apocenter (farthest point) of their orbits. At apocenter, stars are moving slower than their guiding center's circular velocity)

Why is this not symmetric? Because there are more stars in A than B. (density differences)

Two body relaxation

Two-Body Relaxation is the process by which individual stars in a cluster randomly interact with each other through gravity, gradually exchanging ENERGY. Kinetic energy being transferred

Heavy Stars (The "Sinking" Stones): To match the average energy, massive stars must move **slower**. They lose kinetic energy to the lighter stars during interactions. As they lose energy, they fall deeper into the cluster's gravitational potential well.

Light Stars (The "Floating" Feathers): To match the average energy, low-mass stars must move faster. They gain kinetic energy from the heavy stars. This extra speed flings them out to the edges of the cluster.

Evaporation refers to stars escaping a cluster due to gravitational encounters. => **Stellar halo star cluster destruction**

Mass segregation is heavier stars sinking to the center: **heavier mass in globular clusters**

Time scales

Crossing Time (t_{cross}): How long it takes a star to cross the cluster once ($\sim 10^6$ years).

Relaxation Time (t_{relax}): How long it takes for orbits to be randomized ($\sim 10^8$ years).

Evaporation Time (t_{evap}): How long the cluster survives ($\sim 10^{10}$ years).

Jeans Equations

Collisionless Boltzmann Equation
Collisionless boltzmann is about conserving phase space distributions trajectories.

You can get the various moments by integrating the equations against varying cross terms of velocity to get the jeans equations.

They are effectively the same as the fluid euler equation EXCEPT the pressure term is generalized to be anisotropic.

In Class Problems: Applications of Jeans Equations

1. A globular cluster has a stellar density profile $n(r) = n_0\,(1+\frac{r^2}{a^2})^{-5/2}$

   The core radius is $r_c = 2$ pc, and the radial velocity dispersion is σ_{rr}=5 km/s.

   Use the spherical Jeans equation to set up an equation for the mass enclosed within 5 pc.

$$\frac{d}{dr}\left(n_\star \sigma_{rr}^2\right) + 2\beta \frac{n_\star \sigma_{rr}^2}{r} = -n_\star \frac{d\Phi}{dr}. \qquad \beta(r) \equiv 1 - \frac{\sigma_{\theta\theta}^2 + \sigma_{\phi\phi}^2}{2\sigma_{rr}^2} = 1 - \frac{\sigma_t^2}{\sigma_{rr}^2}.$$

Hint: assume σ constant with radius, assume isotropic velocity dispersions, spherical symmetry.

2. In the solar neighborhood, K dwarf stars have a constant vertical velocity dispersion (σ_{zz}=20 km/s) up to |z|<1 kpc. The vertical density profile is exponential, $n(z) = n_0 \exp(-|z|/h)$ with h=300 pc.

Use the axisymmetric Jeans equation (below) and Poisson's equation to compute the surface mass density in the solar neighborhood. Compare your answer to the total baryon mass density, estimated to be 47 $M_\odot$/pc^{-2}.

Hints: 1. You may assume σ_{RZ}^2 is small

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R n_\star \sigma_{Rz}^2\right) + \frac{\partial}{\partial z}\left(n_\star \sigma_{zz}^2\right) = -n_\star \frac{\partial \Phi}{\partial z}.$$

2. You may assume σ_{zz} is constant with |z|

3. Cylindrical coords + axisymmetry

$$4\pi G\rho(R,z) = \nabla^2\Phi(R,z) = \frac{\partial^2\Phi}{\partial z^2} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right).$$

you should find $2\pi G\Sigma(<z) \equiv 2\pi G \int_{-z}^{z} \rho(z')\mathrm{d}z' \approx -\frac{1}{n(z)}\frac{\mathrm{d}}{\mathrm{d}z}\left[n(z)\sigma_z^2\right].$

G=4.3x10^{-3} pc (km/s)^2 $M_\odot^{-1}$

Simplify and apply newtons laws!

① $n = n_0\left(1+\frac{r^2}{a^2}\right)^{-5/2}$, $\sigma = 5\,\mathrm{km/s}$

Jeans equation

$$\frac{d}{dr}(n\sigma_{rr}^2) + 2\beta(r)\frac{\sigma_r^2}{r} \qquad \beta(r) = 1 - \frac{\sigma_{\theta\theta}^2 + \sigma_{\phi\phi}^2}{2\sigma_{rr}^2} = 1 - \frac{\sigma_r^2}{\sigma_{rr}^2}$$

$\downarrow$ simplify

$$\sigma^2\frac{dn}{dr} = n\frac{d\phi}{dr}$$

$\downarrow$ newtons

$$\frac{d\phi}{dr} = \frac{Gm(<r)}{r^2}$$

$$\sigma^2\frac{dn}{dr} = n\frac{Gm}{r}$$

$$M(r) = -\frac{r^2\sigma^2}{Gn}\frac{dn}{dr}$$

$\downarrow$ computer